JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **37**, No. 4, November 2024
http://dx.doi.org/10.14403/jcms.2024.37.4.167

A NOTION OF EQUIVALENCE FOR STURM-LIOUVILLE OPERATORS

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Abstract. In this paper we introduce a notion of equivalence for Sturm Liouville operators in the sense of Pearson equation. As a main result, we prove that a given Sturm Liouville operator and a real-valued polynomial of degree at least one of this Sturm Liouville operator have the same Pearson equation.

1. Introduction

Throughout this paper, we use the standard notation $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}_0$, $\mathcal{P}_n$ denotes the class of all algebraic polynomials of degree exactly n. The set of all probability distributions on $\mathbb{R}$ with finite moments of all orders is denoted by $\text{Prob}_\infty(\mathbb{R})$, i.e., for all $\mu \in \text{Prob}_\infty(\mathbb{R})$, we have

$$\mu_k := \int_{\mathbb{R}} x^k \mu(dx) < \infty, \quad \forall k \in \mathbb{N}_0.$$

Let X be a real–valued random variable with the probability distribution $\mu := \mu_X \in \text{Prob}_\infty(\mathbb{R})$. It is known from [4, 13] that there exists an orthogonal system $(\Phi_n)_{n\in\mathbb{N}_0}$ of monic orthogonal polynomials in $L^2(\mathbb{R}, \mu)$, with $\Phi_0 = 1$, associated with Jacobi sequences $(\omega_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ and $(\alpha_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{R}$, such that

$$(x - \alpha_n)\Phi_n(x) = \Phi_{n+1}(x) + \omega_n\Phi_{n-1}(x), \quad \forall n \in \mathbb{N}_0$$

with convention $\omega_0\Phi_{-1} = 0$, and

$$\langle \Phi_m, \Phi_n \rangle_{L^2(\mathbb{R},\mu)} := \int_{\mathbb{R}} \overline{\Phi_m(x)}\Phi_n(x)\mu(dx) = \omega_1 \cdots \omega_n \delta_{mn}, \ \forall (m, n) \in \mathbb{N}_0^2,$$

Received October 18, 2024; Accepted November 30, 2024.
2020 Mathematics Subject Classification: Primary 34B24; Secondary 47E05.
Key words and phrases: orthogonal polynomial, Sturm-Liouville operator, Pearson equation.
**This paper was supported by a Basic Science Research Program through the NRF funded by the MEST (NRF-2022R1F1A1067601).

where δ_{mn} is the Kronecker delta. Denote by Γ_X^0 the subspace of $L^2(\mathbb{R}, \mu)$ spanned by monic orthogonal polynomials $(\Phi_n)_{n\in\mathbb{N}_0}$. Krall [9] proved that, on the orthogonal gradation

$$\Gamma_X^0 \equiv \bigoplus_{n\in\mathbb{N}_0} \mathcal{P}_n = \bigoplus_{n\in\mathbb{N}_0} \mathbb{C} \cdot \Phi_n \tag{1.1}$$

and under certain conditions on $\mu \in \mathrm{Prob}_\infty(\mathbb{R})$ and its associated moments $(\mu_n)_{n\in\mathbb{N}_0}$, it is possible to construct a differential operator L_N of order N such that

$$L_N \Phi_n = \lambda_n \Phi_n, \quad \forall n \in \mathbb{N}_0, \tag{1.2}$$

where $(\lambda_n)_{n\in\mathbb{N}_0}$ are eigenvalues. Recently, the differential operator L_N has been studied within the quantum language by Accardi *et al.* [2], Dutta *et al.* [5] and Ji [7] following the bridge established by Accardi and Bożejko [1] between orthogonal polynomial theory and the notion of one-mode interacting Fock space, which becomes a fundamental tool in the study of quantum theory.

The aim of this paper is to study the differential operator L_N in the quantum sense started in [2, 5, 7] (see also references therein). Section 2 provides an overview of Sturm-Liouville operators. In Section 3, an equivalence notion on the class of Sturm Louiville operators, namely, equivalence in the sense of Pearson equation is introduced. As a main result in this paper, we prove that a Sturm-Liouville operator L_N and $P(L_N)$, where P has degree at least one, are equivalent in the sense of Pearson equation (see Theorem 3.3).

The notion of equivalence can be applied to the problem exhibited in [2] and relative to the characterization of real-valued random variable of finite type.

2. Sturm Liouville operator

In this paper, unless otherwise specified, all probability distributions are elements of $\mathrm{Prob}_\infty(\mathbb{R})$ and having a probability density function f_X.

2.1. Differential operator of finite order

In [12], Stan *et al.* have introduced the notion of faithfulness for a gradation in the multi–dimensional case (see Definition 4.2 in [12]). In our context, we will restrict their approach to one dimensional case and orthogonal gradation (1.1).

Definition 2.1. Let $k \in \mathbb{Z}$ be a fixed integer. A linear operator T acting on Γ_X^0 is said to be k-faithful to the gradation $(\mathcal{P}_n)_{n\in\mathbb{N}_0}$ if, for every $n \in \mathbb{N}_0$,

$$T\mathcal{P}_n \subseteq \mathcal{P}_{n+k},$$

with the convention that $\mathcal{P}_m := \{0\}$ for all $m < 0$.

The following result, due to Stan *et al.* [12], plays a fundamental role in the representation of operator L_N, described as in (1.2), in terms of differential operators with polynomial coefficients (see [2]).

Theorem 2.2 (Theorem 4.3 in [12]). *If a linear operator T, acting on Γ_X^0, is k–faithful then there exists a unique sequence of complex-valued polynomials $(p_n)_{n\in\mathbb{N}_0}$, with*

$$\deg(p_n) \leq n + k, \quad \forall n \in \mathbb{N}_0,$$

such that for all $g \in \Gamma_X^0$, we have

$$(Tg)(x) := \sum_{n\in\mathbb{N}_0} p_n(x)\partial_x^n g(x) = \sum_{n\in\mathbb{N}_0} p_n(x) g^{(n)}(x),$$

where $g^{(n)}$ denotes the n–th derivative of g. In other words, on Γ_X^0, we can write

$$T = \sum_{n\in\mathbb{N}_0} p_n(x)\partial_x^n. \tag{2.1}$$

Such an operator T, given as in (2.1), *is called a differential operator.*

The following definition was introduced by Accardi *et al.* in [2].

Definition 2.3. A differential operator T acting on Γ_X^0 is said to be of finite order if there exists an integer $N \in \mathbb{N}$ such that

$$T = \sum_{k=0}^{N} p_k(x)\partial_x^k \quad \text{with} \quad p_N \neq 0. \tag{2.2}$$

Such a T, described as in (2.2), is called a differential operator of order N. The algebra of all finite order differential operators acting on Γ_X^0 is denoted by $\mathcal{P}[x, \partial_x]$. In (2.2), if $N = \infty$ then T is called differential operator of infinite order and we denote by $\mathcal{P}[[x, \partial_x]]$ the algebra consisting of all differential operators of infinite order.

2.2. Differential representation induced by smoothness of density

In this section, we will exhibit some properties induced by differential operator ∂_x acting on Γ_X^0.

PROPOSITION 2.4. [2, Lemma 3.4] *Let f_X be a probability density function defined on (a,b). Then*

$$\partial_x^* = \beta(x) - \partial_x,$$

where

$$\beta(x) := -(\log(f_X(x)))' \tag{2.3}$$

and ∂_x^* is interpreted as the quadratic form defined on the set of pairs of differentiable functions $\xi, \eta \in L^2(\mathbb{R}, f_X(x)dx)$ satisfying

$$\overline{\xi}(x)\eta(x)f_X(x)\Big|_a^b := \lim_{x\uparrow b}\overline{\xi}(x)\eta(x)f_X(x) - \lim_{x\downarrow a}\overline{\xi}(x)\eta(x)f_X(x) = 0. \tag{2.4}$$

From Proposition 2.4, it is clear that if f_X is differentiable on (a,b) such that

$$\beta(x) = -(\log(f_X(x)))' = -\frac{(f_X(x))'}{f_X(x)}$$

is a polynomial and satisfies the boundary conditions

$$\lim_{x\downarrow a} p(x)f_X(x) = \lim_{x\uparrow b} p(x)f_X(x) = 0, \tag{2.5}$$

then $(\partial_x)^*$ is a well-defined operator, as adjoint of ∂_x, acting on Γ_X^0. Furthermore, if $f_X \in C^\infty(a,b)$, then for any differential operator $T = \sum_{k=0}^N t_k(x)\partial_x^k \in \mathcal{P}[x,\partial_x]$ acting on Γ_X^0, the adjoint T^* is well-defined as differential operator in $\mathcal{P}[x,\partial_x]$ and acting on Γ_X^0.

In the rest of paper, we assume that the probability density function f_X is a C^∞-function on (a,b) and satisfies the boundary conditions (2.5).

2.3. Definition and some properties

In this section, we revisit the (generalized) Sturm-Liouville operator and some key results related to it discussed by Ji [7] and by Accardi *et al.* [2]. We begin by recalling the definition of a Sturm-Liouville operator.

Recall that, for any $a \in \mathbb{C}$, $(a)_j$ denotes j–th falling factorial of a defined by

$$(a)_0 := 1, \quad (a)_j := a(a-1)\cdots(a-j+1), \quad \forall j \in \mathbb{N}.$$

Definition 2.5. Let X be a real-valued random variable and let f_X be its probability density function on (a,b). We denote by $(\Phi_n)_{n\in\mathbb{N}_0}$ the orthogonal system of monic orthogonal polynomials with respect to f_X. An N–th order differential operator given by

$$L_N = \sum_{k=1}^{N} \sigma_k(x)\partial_x^k,$$

where $\sigma_k(x) = \sum_{j=0}^{k} \sigma_{k,j}x^j$ is real-valued polynomial of degree at most $k \in \{1,...,N\}$, with $\sigma_N \neq 0$, is called an N–th order Sturm Liouville operator associated with f_X if L_N is self-adjoint and satisfies

$$L_N\Phi_n = \lambda_n\Phi_n, \quad \lambda_n = \sum_{j=1}^{N} \sigma_{j,j}(n)_j, \tag{2.6}$$

with convention that $\sum_{j=1}^{N} \sigma_{j,j}^2 \neq 0$.

The following result establishes the connection between Sturm Liouville operator, acting on Γ_X^0, and probability density function f_X of real-valued random variable X.

Theorem 2.6. *Let X be a real-valued random variable with the probability density function f_X on (a,b). Consider an operator H acting Γ_X^0 given by*

$$H = \sum_{k=0}^{N} \sigma_k(x)\partial_x^k \in \mathcal{P}[x,\partial_x].$$

Then we have

(2.7)

$$\begin{aligned} H^* &:= \sum_{k=0}^{N}(-1)^k \left(\frac{(f_X(x))'}{f_X(x)} + \partial_x\right)^k \sigma_k(x) \\ &= \sum_{k=0}^{N}\left(\sum_{m=k}^{N}(-1)^m \sum_{j_1+j_2+k=m} \frac{m!}{j_1!j_2!k!}\,(\sigma_m(x))^{(j_1)}\,\frac{(f_X(x))^{(j_2)}}{f_X(x)}\right)\partial_x^k, \end{aligned}$$

and then $H = H^*$ (i.e., H is Hermitian) on Γ_X^0 if and only if $N \in 2\mathbb{N}$ and, for all $k \in \{0,1,...,N\}$, it holds that

$$\sum_{m=k}^{N}(-1)^m \sum_{j_1+j_2+k=m} \frac{m!}{j_1!j_2!k!}\,(\sigma_m(x))^{(j_1)}\,\frac{(f_X(x))^{(j_2)}}{f_X(x)} = \sigma_k(x). \tag{2.8}$$

Proof. For the explicit proof, see [7]. □

In (2.8), by taking $k = N - 1$, we see that f_X is a solution of the following differential equation (see [5]):

$$N\sigma_N(x)y' + (N(\sigma_N(x))' - 2\sigma_{N-1}(x))y = 0. \tag{2.9}$$

The equation (2.9) is sometimes called a Pearson equation (of type N) associated with the Sturm Liouville operator L_N.

By solving the Pearson equation (2.9), without considering the boundary conditions (2.5), we find that the probability density function f_X associated with L_N is given by

$$f_X(x) = \frac{C}{\sigma_N(x)} \exp\left\{ \frac{2}{N} \int^x \frac{\sigma_{N-1}(s)}{\sigma_N(s)} ds \right\}, \quad \forall x \in (a, b), \tag{2.10}$$

where C is the normalization constant.

In the following examples we list two important examples of second order Sturm Liouvile operator extensively studied by Accardi *et al.* [2].

Example 2.7. Suppose that f_X is the probability density function of a standard Gaussian random variable X given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \forall x \in \mathbb{R}. \tag{2.11}$$

As shown in [2], the Sturm-Liouville operator associated with the standard Gaussian variable X and its eigenvalues $(\lambda_n)_{n\in\mathbb{N}_0}$ are given by

$$L_2 = x\partial_x - \partial_x^2, \quad \lambda_n = n, \quad \forall n \in \mathbb{N}_0. \tag{2.12}$$

Example 2.8. If f_X is the probability density function of a gamma random variable X given by

$$f_X(x) \equiv \gamma_\alpha(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad \alpha > 0, \ \forall x \in (0, +\infty), \tag{2.13}$$

then as shown in [2], the Sturm-Liouville operator associated with the gamma variable X and its eigenvalues $(\lambda_n)_{n\in\mathbb{N}_0}$ are given by

$$L_{2,\gamma_\alpha} = (x - \alpha)\partial_x - x\partial_x^2, \quad \lambda_n = n, \quad \forall n \in \mathbb{N}_0. \tag{2.14}$$

3. An Equivalence of Sturm Liouville operators

Let L_2 be the second order Sturm Liouville operator, given as in (2.12), associated with the standard Gaussian probability density function on $\mathbb{R}$. Taking the square of L_2, we have

$$(L_2)^2 = x(1 + x\partial_x)\partial_x - x\partial_x^3 - (2\partial_x + x\partial_x^2)\partial_x + \partial_x^4. \tag{3.1}$$

Moreover, denoting f the probability density function associated with $(L_2)^2$, we have, from (2.10), that

$$f(x) = C \exp\left\{ \frac{2}{4} \int_0^x (-2s)\, ds \right\} = Ce^{-\frac{x^2}{2}},$$

which is the probability density function of a standard Gaussian distribution, given as in (2.11).

This observation, along with the identity given in (2.9), motivate the following definition.

Definition 3.1. Let $L_M = \sum_{j=1}^{M} \tau_j(x)\partial_x^j$ and $L_N = \sum_{k=1}^{N} \sigma_k(x)\partial_x^k$ be two Sturm Liouville operators of orders M and N, respectively. We say that L_N and L_M are equivalent in the sense of Pearson equation (or simply, equivalent) if

$$\mathbf{P}_N = Q(x)\mathbf{P}_M \quad \text{or} \quad \mathbf{P}_M = Q(x)\mathbf{P}_N$$

for some non-zero real-valued polynomial Q, where

$$\begin{aligned} \mathbf{P}_N f(x) &= N\sigma_N(x)f'(x) + (N\sigma_N(x)' - 2\sigma_{N-1}(x))f(x), \\ \mathbf{P}_M f(x) &= M\tau_M(x)f'(x) + (M\tau_M(x)' - 2\tau_{M-1}(x))f(x). \end{aligned}$$

The following lemma generalizes the discussion mentioned at the beginning of this section.

Lemma 3.2. *Let L_N be an N-th order Sturm Liouville operator. Then, for any $m \in \mathbb{N}$, $(L_N)^m$ is equivalent to L_N.*

Proof. Let $L_N = \sum_{k=1}^{N} \sigma_k(x)\partial_x^k$ be a N-th order Sturm Liouville operator. The statement is trivial for $m = 1$. Now, let $m \geq 2$ be an integer. By applying the Leibniz rule, we obtain that

$$\begin{aligned} (L_N)^m &= \left(\sum_{k_1=0}^{N} \sigma_{k_1}(x)\partial_x^{k_1} \right) \cdots \left(\sum_{k_m=0}^{N} \sigma_{k_m}(x)\partial_x^{k_m} \right) \\ &= \sum_{k_1=0}^{N} \cdots \sum_{k_m=0}^{N} \sigma_{k_1}(x)\partial_x^{k_1}\sigma_{k_2}(x)\partial_x^{k_2} \cdots \partial_x^{k_{m-1}}\sigma_{k_m}(x)\partial_x^{k_m} \\ &= \sum_{k_1=0}^{N} \cdots \sum_{k_m=0}^{N} \sum_{j_1=0}^{k_1} \cdots \sum_{j_{m-1}=0}^{k_{m-1}+\cdots+k_1-j_1-\cdots-j_{m-2}} \\ &\quad \binom{k_1}{j_1} \cdots \binom{k_{m-1}+\cdots+k_1-j_1-\cdots-j_{m-2}}{j_{m-1}} \end{aligned}$$

$$\times \sigma_{k_1}(x) \left(\sigma_{k_2}(x)\right)^{(j_1)} \cdots \left(\sigma_{k_m}(x)\right)^{(j_{m-1})}$$
$$\times \partial_x^{k_m+k_{m-1}+\cdots+k_1-j_1-\cdots-j_{m-2}-j_{m-1}},$$

from which we obtain that

- the order of $(L_N)^m$ is obtained by taking the $(2m-1)$-tuple
$$(k_1, ..., k_m, j_1, ..., j_{m-1}) = (N, ..., N, 0, ..., 0),$$
i.e., $(L_N)^m$ is a differential operator of order mN and $(\sigma_N)^m$ is the coefficient polynomial associated with ∂_x^{mN}.
- the coefficient polynomial associated with ∂_x^{mN-1} is obtained by taking the $(2m-1)$-tuples $(k_1, ..., k_m, j_1, ..., j_{m-1})$ with the values
$$(N-1, ..., N, 0, ..., 0), \cdots, (N, ..., N-1, 0, ..., 0),$$
$$(N, ..., N, 1, ..., 0), \cdots, (N, ..., N, 0, ..., 1).$$

Therefore, the polynomial coefficient associated with ∂_x^{mN-1} is given by

$$m\sigma_{N-1}(x) \left(\sigma_N(x)\right)^{m-1} + \frac{m(m-1)}{2} N \left(\sigma_N(x)\right)' \sigma_N(x).$$

It follows that $(L_N)^m$ can be re–written as follows:

$$(L_N)^m = A + R(x)\partial_x^{mN-1} + \left(\sigma_N(x)\right)^m \partial_x^{mN},$$

where A is a differential operator of order at most $mN-2$ and

$$R(x) := m\sigma_{N-1}(x) \left(\sigma_N(x)\right)^{m-1} + \frac{m(m-1)}{2} N \left(\sigma_N(x)\right)' \sigma_N(x).$$

Therefore, by applying (2.9)and constructing the Pearson equation associated with the Sturm Liouville operator $(L_N)^m$, it follows that

$$\begin{aligned}
&mN \left(\sigma_N(x)\right)^m y' + \left(mN \left(\left(\sigma_N(x)\right)^m\right)' - 2R(x)\right) y \\
&\quad = mN \left(\sigma_N(x)\right)^m y' + \left(mN \left(m \left(\sigma_N(x)\right)' \left(\sigma_N(x)\right)^{m-1}\right) - 2R(x)\right) y \\
&\quad = m \left(\sigma_N(x)\right)^{m-1} \Big(N\sigma_N(x)y' + \Big(mN \left(\sigma_N(x)\right)' \\
&\qquad -2\left(\sigma_{N-1}(x) + \frac{(m-1)}{2} N \left(\sigma_N(x)\right)'\right)\Big) y\Big) \\
&\quad = m \left(\sigma_N(x)\right)^{m-1} \left(N\sigma_N(x)y' + \left(N \left(\sigma_N(x)\right)' - 2\sigma_{N-1}(x)\right) y\right),
\end{aligned}$$

which means that $(L_N)^m$ and L_N are equivalent. □

Notice that the Lemma 3.2 can be generalized as follows

THEOREM 3.3. *If L_N is an N-th order Sturm Liouville operator, then, for any real-valued polynomial P of degree $m \geq 1$, $P(L_N)$ and L_N are equivalent.*

Proof. Let L_N be an N-th order Sturm Liouville operator given by

$$L_N = \sum_{k=1}^{N} \sigma_k(x)\partial_x^k, \quad \sigma_N(x) \neq 0,$$

where $\sigma_k(x)$ is a real-valued polynomial of degree at most $k \in \{1, ..., N\}$, and let $P(x) = \sum_{j=0}^{m} c_j x^j$ be a real-valued polynomial of degree $m \geq 1$, that is, c_m is non–zero. Then it is obvious that $P(L_N)$ is a differential operator of order mN and we have

$$\begin{aligned} P(L_N) &= \sum_{k=0}^{m} c_j \left(\sum_{k=0}^{N} \sigma_k(x)\partial_x^k \right)^j \\ &= \sum_{k=0}^{m-1} c_j \left(\sum_{k=0}^{N} \sigma_k(x)\partial_x^k \right)^j + c_m \left(\sum_{k=0}^{N} \sigma_k(x)\partial_x^k \right)^m . \end{aligned}$$

Notice the follows:

- $\sum_{k=0}^{m-1} c_j \left(\sum_{k=0}^{N} \sigma_k(x)\partial_x^k \right)^j$ is a differential operator of order at most $(m-1)N$,
- $c_m \left(\sum_{k=0}^{N} \sigma_k(x)\partial_x^k \right)^m$ is a differential operator of order mN (because $c_m \neq 0$).

Moreover, $mN - 1 > (m-1)N$ (because $N \geq 2$) and the Pearson equation of order mN, associated with $P(L_N)$ is the Pearson equation associated with $c_m(L_N)^m$. By using Lemma 3.2, we can conclude $P(L_N)$ is Pearson equivalent to L_N. □

Now, we may ask whether it is possible to find two equivalent Sturm Liouville operators S and T such that their corresponding probability density functions are different. The next example gives a positive answer to this question.

Example 3.4. Let $R > 0$ be given. Consider the fourth order differential operator L_4 given by

$$L_4 = \sum_{k=1}^{4} h_k(x)\partial_x^k, \tag{3.2}$$

where

$$\begin{aligned} h_1(x) &= 2(R+1)x - 2R, \quad h_2(x) = x^2 - 2(R+3)x, \\ h_3(x) &= -2x(x-2), \quad h_4(x) = x^2. \end{aligned} \tag{3.3}$$

It is know from Section II.2 in [11] that L_4, given as in (3.2), is a Sturm Liouville operator associated with the so-called *Laguerre–type orthogonal polynomials*, with eigenvalues $(\lambda_n)_{n\in\mathbb{N}_0}$ given by

$$\lambda_n = n(n+2R+1), \quad \forall n \in \mathbb{N}_0.$$

Now, on one hand, using (2.9), the Pearson equation associated with L_4, given as in (3.2), becomes

$$0 = 4x^2y' + (4(2x) + 2(2x^2 - 4x))y = 4x^2\left(y' + y\right). \tag{3.4}$$

On the other hand, consider the Sturm Liouville operator $L_2 \equiv L_{2,\gamma_1}$, given as in (2.14) (with $\alpha = 1$), associated with probability distribution γ_1, , i.e.

$$L_{2,\gamma_1} = (x-1)\partial_x - x\partial_x^2. \tag{3.5}$$

It follows, from (2.9), that the Pearson equation associated with L_{2,γ_1} is given by

$$0 = -2xy' + (2(-1) - 2(x-1))y = -2x(y + y'). \tag{3.6}$$

Moreover, from (3.4) and (3.6), we have

$$4x^2\left(y' + y\right) = -2x\left(-2x(y + y')\right), \quad \forall x \in (0, +\infty).$$

Therefore, L_4 and L_{2,γ_1} are equivalent in the sense of the Pearson equation.

Remark 3.5. In Example 3.4, even though the Sturm Liouville operators L_4 and L_{2,γ_1} are equivalent, we should point out that the probability density function γ_1, associated with L_{2,γ_1}, is not associated with L_4. In fact, if γ_1 were associated with L_4, it would satisfy the equation

$$\sum_{m=k}^{4}(-1)^m \sum_{j_1+j_2+k=m} \frac{m!}{j_1!j_2!k!}\left(h_m(x)\right)^{(j_1)} \frac{\left(\gamma_1(x)\right)^{(j_2)}}{\gamma_1(x)} = h_k(x) \tag{3.7}$$

for all $k \in \{1,2,3,4\}$, where

$$\gamma_1(x) = e^{-x}, \quad \forall x \in (0,\infty)$$

Let us try to verify (3.7) for $k = 2$. In fact, using the definition of $h_k(x)$ in (3.3), the left hand side of (3.7) becomes

$$\sum_{m=2}^{4}(-1)^m \sum_{j_1+j_2=m-2} \frac{m!}{j_1!j_2!2!}\left(h_m(x)\right)^{(j_1)} \frac{\left(\gamma_1(x)\right)^{(j_2)}}{\gamma_1(x)}$$

$$= \sum_{m=2}^{4}(-1)^m \sum_{j_1+j_2=m-2} (-1)^{j_2}\frac{m!}{j_1!j_2!2!}\left(h_m(x)\right)^{(j_1)}$$

$$
\begin{aligned}
&= \sum_{j_1+j_2=0} (-1)^{j_2} \frac{2!}{j_1! j_2! 2!} (h_2(x))^{(j_1)} - \sum_{j_1+j_2=1} (-1)^{j_2} \frac{3!}{j_1! j_2! 2!} (h_3(x))^{(j_1)} \\
&\quad + \sum_{j_1+j_2=2} (-1)^{j_2} \frac{4!}{j_1! j_2! 2!} (h_4(x))^{(j_1)} \\
&= h_2(x) - 3\big(h_3(x)' - h_3(x)\big) + \big(h_4(x) + h_4(x)'' - 12h_4(x)'\big) \\
&= x^2 - 2(R+3)x - 3\big\{2x^2 - 8x + 4\big\} + \big(x^2 + 2 - 24x\big) \\
&= -4x^2 - 2(R+3)x - 10,
\end{aligned}
$$

which is not equal to $h_2(x)$, defined in (3.3). Therefore γ_1 is not associated with L_4 given as in (3.2). Moreover, there is no real-valued polynomial P such that $L_4 = P(L_{2,\gamma_1})$. In deed, suppose that there exists a real-valued polynomial P such $L_4 = P(L_{2,\gamma_1})$. Since L_{2,γ_1} is a second order Sturm-Liouville operator, $P(x)$ must be of the form $P(x) = \alpha x^2 + \beta x$, where α, β are reals. By (3.5) and direct computation, for any polynomial $Q(x)$, we have

$$
\begin{aligned}
(L_{2,\gamma})^2 Q(x) = (x-1)Q'(x) + (x^2 - 5x + 2)Q''(x) \\
- 2x(x-2)Q^{(3)}(x) + x^2 Q^{(4)}(x).
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
P(L_{2,\gamma}) &= \alpha\,(L_{2,\gamma})^2 + \beta\,(L_{2,\gamma}) \\
&= (x-1)(\alpha+\beta)\partial_x + (\alpha x^2 - (5\alpha+\beta)x + 2\alpha)\partial_x^2 \\
&\quad - 2x\alpha(x-2)\partial_x^3 + \partial_x^4.
\end{aligned} \tag{3.8}
$$

Since $P(L_{2,\gamma}) = L_4$, it follows from (3.2), (3.3) and (3.8) that

$$
\begin{aligned}
2(R+1)x - 2R &= (x-1)(\alpha+\beta), \\
x^2 - 2(R+3)x &= \alpha x^2 - (5\alpha+\beta)x + 2\alpha, \\
-2x(x-2) &= -2x\alpha(x-2), \\
x^2 &= \alpha x^2,
\end{aligned}
$$

which implies that $\alpha = 1$ (from the last identity) and $\alpha = 0$ (from the second identity). This is a contradiction. Therefore, it is impossible to construct a real-valued polynomial P, such that $P(L_{2,\gamma_1}) = L_4$.

Remark 3.6. H. L. Krall proved in [9] that L_4, given as in (3.2), is self-adjoint with respect to the probability density function f_X given by (see Example 2.2 in [10])

$$f_X(x) = C\left[\frac{1}{R}\delta(x) + e^{-x}H(x)\right], \quad \forall x \in [0, +\infty), \tag{3.9}$$

where δ is the Dirac distribution (also called Dirac delta function) and H is the Heaviside function defined by

$$H(x) := \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Notice that the probability density function f_X, given as in (3.9), is not differentiable in the classical sense, but it is differentiable in distribution sense.

PROPOSITION 3.7. *Let L_N be an N-th order Sturm Liouville operator associated with a probability density function $f_X \in C^\infty(a,b)$. Then for any real-valued polynomial P, $P(L_N)$ is associated with f_X.*

Proof. Since $L_N^* = L_N$ with respect to f_X, for any $m \in \mathbb{N}$, it holds that

$$[(L_N)^m]^* = [(L_N)^*]^m = (L_N)^m.$$

This implies that for any real-valued polynomial P, $P(L_N)^* = P(L_N)$. Let $(\Phi_n)_{n\in\mathbb{N}_0}$ be the orthogonal system of monic orthogonal polynomials with respect f_X satisfying the eigenvalue problem given as in (2.6). Then for any $n \in \mathbb{N}_0$, it follows from applying $P(L_N)$ to (2.6) that

$$P(L_N)\Phi_n = P(\lambda_n)\Phi_n.$$

Therefore, $P(L_N)$ is a Sturm-Liuoville operator associated with f_X. □

REMARK 3.8. Motivated by Proposition 3.7, some relations between two Sturm Liouville operators which are associated with same probability density function will be discussed in a separated paper.

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